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Degree of Simultaneous Approximation of Bivariate Functions by Gordon Operators

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We investigate the degree of approximation of bivariate functions on a rectangle by various (discrete) spline-blended operators. Our aim is to give a fuller description than is available in the literature by using mixed moduli of smoothness of higher orders. The crucial tool from the univariate case is a generalization of a theorem of Sharma and Meir on the degree of simultaneous approximation by cubic spline interpolators. The main results for the multivariate case are two theorems expressing certain permanence principles, which explain how the Boolean sums and certain (discrete) blending operators inherit quantitative properties from their univariate building blocks. Various historical remarks and numerous references are included in order to draw the reader's attention to the somewhat diverse history of the subject.

1. INTRODUCTION

The present paper supplements our recent research (see [12, 14, 15]) on the degree of approximation in $C(I\times J)$, the space of real-valued functions which are continuous on the rectangle $I\times J$, where $I := [a, b]$ and $J := [c, d]$ are non-trivial compact intervals of the real axis $\mathbb{R}$. The approximating functions used were trigonometric and algebraic blending functions (pseudopolynomials). Functions of this type were introduced in two papers of Marchaud [31, 32] and are (for the algebraic case) defined by the scheme

$$I\times J \ni (x, y) \mapsto \sum_{i=0}^{n} x^i \cdot A_i(y) + \sum_{j=0}^{m} B_j(x)\cdot y^j \in \mathbb{R}.$$

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Here A_i and B_j are bounded, real-valued functions on J and I, respectively, and $n, m \geqslant 0$ are integers.

Thus the use of the so-called *blending approach* to construct approximating bivariate functions is a classical one. In addition to the work by Marchaud, other early papers on the subject were written by Neder [36, 37], Mangeron [30], Popoviciu [39, 40], and by Nicolescu [38]. These historical facts seem to have been overlooked for several years, although Gordon's highly original paper [19] is a most valuable source of historical references, mentioning also Stancu's [46] and Coons' [10] important work on the subject. The relevance of this background was emphasized in Birkhoff's paper [5] on algebraic aspects of multivariate interpolation.

While Gordon [16–22] carried out most of his work on the Boolean sum method in the late 60's, there were three schools, two Russian and one Romanian, which independently and almost simultaneously worked on related problems. Unfortunately, they used a somewhat different and sometimes misleading terminology, so that their results remained largely unknown in the Western hemisphere until recently (see, e.g., [7, 41, 1]). More detailed information in regard to the history of the subject and a brief survey of some of the results of the Russian and Romanian schools can be found in [12].

The bases $\{x^0, x^1, ..., x^n\}$ of $\prod_n(I)$ and $\{y^0, y^1, ..., y^m\}$ of $\prod_m(J)$ used in the above representation of a pseudopolynomial may be replaced by bases of other finite-dimensional subspaces of $C(I)$ and $C(J)$. If we choose instead bases of certain spaces of splines defined on I and J, respectively, we are led to *spline-blended operators* (such as, for instance, spline-blended surface interpolators), also known as *Gordon operators*. A more exact definition will be given below. The term "spline-blended operator," or similar expressions, are mainly used in papers dealing with the approximation-theoretical aspects of the subject; the terminology "Gordon operator" or "Gordon surface" can be found in various articles written from the viewpoint of Computer-aided Geometric Design (see, e.g., [6, 2]).

The aim of this note is to carry out an investigation on the degree of approximation by Gordon operators and some of their generalizations and modifications which is analogous to our previous work on approximation by algebraic pseudopolynomials, and thus to give a fuller description of their approximation behavior than is available in the literature. Both pseudopolynomials and spline blended interpolants are the result of applying a certain blending scheme to a bivariate function f given on the square $I \times J$. Another way of saying this is that they arise from some *Boolean sum* of operators applied to certain functions f in a function space F. To be more specific, let us recall that if P and Q are linear operators defined on a function space F (consisting of functions over a domain D) and mapping F into itself, then the Boolean sum of P and Q is given by

$$(P \oplus Q)(f, x) := (P + Q - PQ)(f, x)$$
$$= P(f - Qf, x) + Q(f, x), \qquad f \in F, x \in D.$$

For $D = I \times J$ and $f \in C(I \times J)$, we define the *partial functions* f_x and f^y by

$$f_x(y) := f^y(x) := f(x, y), \qquad \text{for all} \quad (x, y) \in I \times J.$$

If L is an operator given on some space G of functions in the variable $x \in I$, then the *parametric extension* ${}_xL$ of L to all bivariate functions $f: I \times J \to \mathbb{R}$ such that $f^y \in G$ for all $y \in J$ is given by

$${}_xL(f; x, y) := L(f^y; x), \qquad (x, y) \in I \times J.$$

Likewise, if M is an operator defined for certain functions in the variable $y \in J$, the parametric extension ${}_yM$ of M is defined by

$${}_yM(f; x, y) := M(f_x; y).$$

It thus makes sense to consider the Boolean sum

$${}_xL \oplus {}_yM : F \to ({}_xL \oplus {}_yM)(F)$$

on the set F of all bivariate functions for which this operator is well-defined.

Before describing further what we will denote as a Gordon operator, we make some notational remarks. For $k, l \in \mathbb{N}_0$, the symbol $D^{(k,l)}$ denotes the partial differential operator $\partial^{k+l}/\partial x^k \partial y^l$; occasionally we will write $f^{(k,l)}$ instead of $D^{(k,l)}f$. We define

$$C^{p,q}(I \times J) := \{f: I \times J \to \mathbb{R} \mid D^{(k,l)}f \text{ is continuous for } 0 \leqslant k \leqslant p, 0 \leqslant l \leqslant q\}.$$

The corresponding symbols used for the univariate case will be $C^p(I)$, $D^{(k)}$, $(d/dx)^k$, and $f^{(k)}$, respectively. For $p = q = 0$, we write $C(I)$ and $C(I \times J)$ instead of $C^0(I)$ and $C^{0,0}(I \times J)$. Similarly, $D^{(0)}$ and $D^{(0,0)}$ mean the identity operators on the appropriate spaces.

For Gordon operators, the setting is now as follows. Let

$$\Delta_n = \Delta_{x,n}: a = x_0 < x_1 < \cdots < x_n = b$$

be a collection of knots, i.e., a mesh, partitioning the interval I of the real line $\mathbb{R}$. We define

$$\Delta x_i := x_{i+1} - x_i, \qquad 0 \leqslant i \leqslant n-1,$$
$$\delta := \max_{0 \leqslant i \leqslant n-1} \Delta x_i \qquad (\text{"mesh gauge"}),$$
$$\beta := \delta / \min_{0 \leqslant i \leqslant n-1} \Delta x_i \qquad (\text{"mesh ratio"}).$$

If

$P^{2m-1}(I, \Delta_n)$

$:= \{p \mid p \text{ is polynomial of degree } 2m-1 \text{ in each interval } (x_i, x_{i+1})\}$,

then

$$S^3(I, \Delta_n) := P^3(I, \Delta_n) \cap C^2(I)$$

constitutes a space of cubic splines.

Given any function f possessing a first derivative at x_0 and x_n, the so-called *Type I cubic spline interpolant* of f ("clamped spline") is by definition the unique element $s_f \in S^3(I, \Delta_n)$ satisfying

$$\begin{aligned} s_f(x_i) &= f(x_i), \qquad 0 \leqslant i \leqslant n, \\ s_f'(x_i) &= f'(x_i), \qquad i = 0, n. \end{aligned}$$

The spline s_f can be written as

$$s_f(x) = f'(x_0) \cdot \Phi_{-1}(x) + \left(\sum_{i=0}^{n} f(x_i) \cdot \Phi_i(x) \right) + f'(x_n) \cdot \Phi_{n+1}(x),$$

where Φ_i, $-1 \leqslant i \leqslant n+1$, are the cardinal splines of Type I interpolation. Clearly, the mapping

$$S_{\Delta_n}: C^1(I) \ni f \mapsto s_f \in S^3(I, \Delta_n) \subset C^2(I)$$

is linear. Using a further partition $\Delta_m = \Delta_{y,m}$ of $J = [c, d]$ yields a second linear spline operator

$$S_{\Delta_m}: C^1(J) \ni f \mapsto s_f \in S^3(J, \Delta_m) \subset C^2(J).$$

Instead of the operator S_{Δ_n}, frequently the so-called *natural spline operators* $T_{\Delta_n}: C(I) \to S^3(I, \Delta_n)$ have been considered. They yield splines satisfying the interpolation conditions

$$\begin{aligned} T_{\Delta_n}(f, x_i) &= f(x_i), \qquad 0 \leqslant i \leqslant n, \\ (T_{\Delta_n} f)''(x_i) &= 0, \qquad i = 0, n. \end{aligned}$$

The two free parameters in a cubic spline interpolant can also be assigned in other ways (periodic splines, Type II cubic spline interpolants, etc.)

In this paper, we will denote as a (special) Gordon operator the Boolean sum

$${}_x S_{\Delta_n} \oplus {}_y S_{\Delta_m},$$

or the operator

$$ {}_xT_{\Delta_n} \oplus {}_yT_{\Delta_m}. $$

From the above representation of $S_{\Delta_n}(f, x) = s_f(x)$, it follows that

$$ {}_xS_{\Delta_n}(f; x, y) = \frac{\partial f}{\partial x}(x_0, y) \cdot \Phi_{-1}(x) + \sum_{i=0}^{n} f(x_i, y) \cdot \Phi_i(x) + \frac{\partial f}{\partial x}(x_n, y) \cdot \Phi_{n+1}(x). $$

Hence, ${}_xS_{\Delta_n} f$ is an element of the space

$$ S^3(I, \Delta_n) \otimes C^1(J), $$

and likewise, ${}_yS_{\Delta_m}$ maps into

$$ C^1(I) \otimes S^3(J, \Delta_m). $$

(We note here that, given the (real) linear spaces V, W of univariate functions, the tensor product $V \otimes W$ of bivariate functions is the linear hull of all product-type functions $f(x) \cdot g(y)$ with $f \in V$ and $g \in W$.)

Furthermore, the product operator

$$ {}_xS_{\Delta_n} \circ {}_yS_{\Delta_m} $$

maps into $S^3(I, \Delta_n) \otimes C^2(J) \cap C^2(I) \otimes S^3(J, \Delta_m)$, so that

$$ {}_xS_{\Delta_n} \oplus {}_yS_{\Delta_m} : C^{1,1}(I \times J) \to S^3(I, \Delta_n) \otimes C^1(J) + C^1(I) \otimes S^3(J, \Delta_m). $$

Analogously,

$$ {}_xT_{\Delta_n} \oplus {}_yT_{\Delta_m} : C(I \times J) \to S^3(I, \Delta_n) \otimes C(J) + C(I) \otimes S^3(J, \Delta_m). $$

Assertions for the degree of approximation of smooth functions by ${}_xS_{\Delta_n} \oplus {}_yS_{\Delta_m}$ and by ${}_xT_{\Delta_n} \oplus {}_yT_{\Delta_m}$ were made by several authors; we mention papers by Gordon [19] and by Carlson and Hall [8], where estimates for $f \in C^{4,4}(I \times J)$ were given. Also, Mettke [33] showed that, under certain assumptions, estimates on simultaneous approximation involving a mixed modulus of order (1, 1) are valid. In this paper we shall, among others, prove a certain extension of the main result of Mettke [33, Satz 1] by giving more complete estimates for simultaneous approximation in $C^{1,1}(I \times J)$ and $C(I \times J)$, respectively, using *mixed moduli of smoothness* of higher orders. For $r, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} := \{1, 2, 3, ...,\} \cup \{0\}$, these moduli of order (r, s) are given for $f \in C(I \times J)$, and $\delta_1, \delta_2 \geqslant 0$ by

$$ \omega_{r,s}(f; \delta_1, \delta_2) := \sup\{|{}_x\Delta_{h_1}^r \circ {}_y\Delta_{h_2}^s f(x, y)| : (x, y), (x + rh_1, y + sh_2) \in I \times J, |h_i| \leqslant \delta_i, i = 1, 2\}. $$

Here,

$$ {}_x\Delta^r_{h_1} \circ {}_y\Delta^s_{h_2} f(x, y) := \sum_{\rho=0}^{r} \sum_{\sigma=0}^{s} (-1)^{r+s-\rho-\sigma} \binom{r}{\rho}\binom{s}{\sigma} f(x+\rho h_1, y+\sigma h_2) $$

is a mixed difference of order (r, s) with increment h_1 with respect to x and increment h_2 with respect to y. Several properties of this mixed modulus of smoothness can be found in Schumaker's book [44]. For the sake of completeness we mention here that the mixed modulus of order $(1, 1)$ was already used in a paper by Munteanu and Schumaker [34], who gave, among others, inequalities for the approximation by Boolean sums of variation diminishing spline operators.

2. Degree of Simultaneous Approximation by Boolean Sums of Parametric Extensions

In addition to the notations introduced above, we shall also use the following: The space $C^p(I)$ (of univariate functions f) will be equipped with the norm

$$ \| f \|_{C^p(I)} := \max\{ \| f^{(k)} \|_\infty : 0 \leqslant k \leqslant p \}; $$

here $\| \cdot \|_\infty$ denotes the Čebyšev norm on I. Clearly, $\| \cdot \|_{C^0(I)} = \| \cdot \|_\infty$.

If L is a continuous linear operator mapping $(C^p(I), \| \cdot \|_{C^p(I)})$ into $(C^{p'}(I), \| \cdot \|_{C^{p'}(I)})$, its operator norm will be denoted by $\| L \|_{[C^p(I), C^{p'}(I)]}$.

Lemma 2.1 [14]. *For $p, q \in \mathbb{N}_0$, let the space $C^{p,q}(I \times J)$ be given as above. Let $M: (C^q(J), \| \cdot \|_{C^q(J)}) \to (C^0(J), \| \cdot \|_{C^0(J)})$ be linear and continuous. Under these assumptions, the following two statements hold:*

(i) *for each fixed $y \in J$ and each $f \in C^{p,q}(I \times J)$, the function*

$$ I \ni x \mapsto M(f_x, y) \in \mathbb{R} $$

is p-times continuously differentiable (with respect to x);

(ii) *for $0 \leqslant k \leqslant p$, we have*

$$ \left(\frac{d}{dx}\right)^k [x \mapsto M(f_x, y)] = [x \mapsto M((f^{(k,0)})_x ; y)], $$

i.e.,

$$ D^{(k,0)} \circ {}_yM = {}_yM \circ D^{(k,0)} \quad \text{on} \quad C^{p,q}(I \times J) \quad \text{for} \quad 0 \leqslant k \leqslant p. $$

(iii) *If* $L: (C^p(I), \|\cdot\|_{C^p(I)}) \to (C^0(I), \|\cdot\|_{C^0(I)})$ *is linear and continuous, then statements akin to* (i) *and* (ii) *hold; in particular,*

$$_xL \circ D^{(0,l)} = D^{(0,l)} \circ {_xL} \quad on \quad C^{p,q}(I \times J) \quad for \quad 0 \leqslant l \leqslant q.$$

The main goal of the following theorem is to describe how quantitative properties of certain univariate operators L and M are inherited by the Boolean sum of their parametric extensions. It thus supplements results of a similar nature which were given by Barnhill and Gregory [3] (see also Schumaker [43], Litvin [29]). The theorem's form as presented here constitutes a slight generalization of the corresponding assertion proved in [14], and is the more suitable one for our present purposes. For the reader's convenience, we include a proof.

THEOREM 2.2 (cf. [14]). *For* $p, p', q, q' \in \mathbb{N}_0$, *let linear operators* $L: C^p(I) \to C^{p'}(I)$ *and* $M: C^q(J) \to C^{q'}(J)$ *be given, such that for fixed* $r, s \in \mathbb{N}_0$ *the following hold:*

(i) $|(g - Lg)^{(k)}(x)| \leqslant \Gamma_{r,k,L}(x) \cdot \omega_r(g^{(p)}; \Lambda_{r,L}(x))$ *for all* $x \in I$, *all* $g \in C^p(I)$, *and all* $0 \leqslant k \leqslant p^* := \min\{p, p'\}$, *and*

(ii) $|(h - Mh)^{(l)}(y)| \leqslant \Gamma_{s,l,M}(y) \cdot \omega_s(h^{(q)}; \Lambda_{s,M}(y))$ *for all* $y \in J$, *all* $h \in C^q(J)$, *and all* $0 \leqslant l \leqslant q^* := \min\{q, q'\}$;

here the Γ's and Λ's are assumed to be bounded real-valued functions. Then for all $(x, y) \in I \times J$, *all* $f \in C^{p,q}(I \times J)$, *and for* $(0, 0) \leqslant (k, l) \leqslant (p^*, q^*)$, *there holds:*

$$|(f - ({_xL} \oplus {_yM})f)^{(k,l)}(x, y)| \leqslant \Gamma_{r,k,L}(x) \cdot \Gamma_{s,l,M}(y) \cdot \omega_{r,s}(f^{(p,q)}; \Lambda_{r,L}(x), \Lambda_{s,M}(y)).$$

Proof. Note that

$$\begin{aligned} &|(f - ({_xL} \oplus {_yM})f)^{(k,l)}(x, y)| \\ &\quad = |D^{(k,l)} \circ (I - ({_xL} \oplus {_yM}))(f; x, y)| \\ &\quad = |D^{(k,0)}(\{D^{(0,l)} \circ (I - {_yM})\}(f) - \{D^{(0,l)} \circ {_xL} \circ (I - {_yM})\}(f))(x, y)|. \end{aligned}$$

By our assumption (i) on L, for $0 \leqslant k \leqslant p^*$ we have

$$\|(Lg)^{(k)}\|_I \leqslant \|g^{(k)}\|_I + c \cdot \|g^{(p)}\|_I \leqslant c^* \cdot \|g\|_{C^p(I)};$$

here c and c^* are suitable numerical constants. Hence

$$\|Lg\|_{C^{p^*}(I)} \leqslant c^* \cdot \|g\|_{C^p(I)},$$

so that

$$L: (C^p(I), \|\cdot\|_{C^p(I)}) \to (C^{p^*}(I), \|\cdot\|_{C^{p^*}(I)})$$

is continuous. Clearly, this is also true if the topology in the range is replaced by the coarser one induced on $C^{p^*}(I)$ by $\|\cdot\|_{C^0(I)}$. Hence

$$L: (C^p(I), \|\cdot\|_{C^p(I)}) \to (C^{p^*}(I), \|\cdot\|_{C^0(I)}) \subset (C^0(I), \|\cdot\|_{C^0(I)})$$

is continuous as well, so that Lemma 2.1 implies

$$D^{(0,l)} \circ {}_xL = {}_xL \circ D^{(0,l)} \qquad \text{for} \quad 0 \leqslant l \leqslant q.$$

Thus

$$\begin{aligned}&|D^{(k,0)}(\{D^{(0,l)} \circ (I - {}_yM)\}(f) - \{D^{(0,l)} \circ {}_xL \circ (I - {}_yM)\}(f))(x, y)|\\ &\quad = |D^{(k,0)}(\{D^{(0,l)} \circ (I - {}_yM)\}(f) - {}_xL\{(D^{(0,l)} \circ (I - {}_yM))(f)\})(x, y)|.\end{aligned}$$

Now the assumption on the quantitative behavior of the univariate operator L may be used, since the function in ( ) can be written as a univariate function of x with parameter y, namely as

$$x \mapsto (\{D^{(0,l)} \circ (I - {}_yM)\}(f))^y(x) - L(\{(D^{(0,l)} \circ (I - {}_yM))(f)\}^y; x).$$

Applying $D^{(k,0)}$ to the function in ( ) is the same as differentiating the latter univariate function with respect to x. Hence, by assumption (i), the quantity which we are interested in is bounded from above by

$$\Gamma_{r,k,L}(x) \cdot \omega_r\left(\left(\frac{d}{dx}\right)^p \{(D^{(0,l)} \circ (I - {}_yM))(f)\}^y; \Lambda_{r,L}(x)\right), \qquad 0 \leqslant k \leqslant p^*.$$

The rth modulus of smoothness may be replaced by

$$\left|{}_x\Delta^r_{\delta^*}\left[\left(\frac{d}{dx}\right)^p \{(D^{(0,l)} \circ (I - {}_yM))(f)\}^y\right](x^*)\right|, \qquad x^* \in I,\ |\delta^*| \leqslant \Lambda_{r,L}(x).$$

Next we investigate the latter quantity by using the information available on M. The absolute value of the rth order difference is equal to

$$\left|\sum_{\rho=0}^{r} (-1)^\rho \binom{r}{\rho}\left[\left(\frac{d}{dx}\right)^p \{(D^{(0,l)} \circ (I - {}_yM))(f)\}^y\right](x^* + \rho \cdot \delta^*)\right|.$$

Because

$$\begin{aligned}\left(\frac{d}{dx}\right)^p &\{(D^{(0,l)} \circ (I - {}_yM))(f)\}^y(x)\\ &= (D^{(0,l)} \circ D^{(p,0)} \circ (I - {}_yM))(f; x, y)\\ &= (D^{(0,l)} \circ (I - {}_yM) \circ D^{(p,0)})(f; x, y)\end{aligned}$$

(where the last equality is a consequence of $D^{(p,0)} \circ {}_yM = {}_yM \circ D^{(p,0)}$), it follows that the rth order difference may be written as

$$\left|\sum_{\rho=0}^{r}(-1)^{\rho}\binom{r}{\rho}\cdot(D^{(0,l)}\circ(I-{}_yM)(D^{(p,0)}f; x^*+\rho\cdot\delta^*, y)\right|$$

$$=\left|\sum_{\rho=0}^{r}(-1)^{\rho}\binom{r}{\rho}\cdot\left\{\left(\frac{d}{dy}\right)^{l}(D^{(p,0)}f)_{x^*+\rho\cdot\delta^*}(y)\right.\right.$$

$$\left.\left.-\left(\frac{d}{dy}\right)^{l}({}_yM\circ D^{(p,0)}(f))_{x^*+\rho\cdot\delta^*}(y)\right\}\right|$$

$$=\left|\sum_{\rho=0}^{r}(-1)^{\rho}\binom{r}{\rho}\cdot\left\{\left(\frac{d}{dy}\right)^{l}(D^{(p,0)}f)_{x^*+\rho\cdot\delta^*}(y)\right.\right.$$

$$\left.\left.-\left(\frac{d}{dy}\right)^{l}M((D^{(p,0)}f)_{x^*+\rho\cdot\delta^*}; y)\right\}\right|$$

$$=\left|\left[\left(\frac{d}{dy}\right)^{l}-\left(\frac{d}{dy}\right)^{l}\circ M\right]\left(\sum_{\rho=0}^{r}(-1)^{\rho}\binom{r}{\rho}\cdot(D^{(p,0)}f)_{x^*+\rho\cdot\delta^*}; y\right)\right|.$$

This difference may now be evaluated by using assumption (ii) on M. Hence for $0\leqslant l\leqslant q^*$, its absolute value is less than or equal to

$$\Gamma_{s,l,M}(y)\cdot\omega_s\left(\left(\frac{d}{dy}\right)^{q}\cdot\sum_{\rho=0}^{r}(-1)^{\rho}\binom{r}{\rho}\cdot(D^{(p,0)}f)_{x^*+\rho\cdot\delta^*}; \Lambda_{s,M}(y)\right).$$

The sth order modulus may be written as

$$\left|{}_y\Delta^{s}_{\eta^*}\left[\left(\frac{d}{dy}\right)^{q}\cdot\sum_{\rho=0}^{r}(-1)^{\rho}\binom{r}{\rho}\cdot(D^{(p,0)}f)_{x^*+\rho\cdot\delta^*}\right](y^*)\right|$$

for some $y^*\in J$ and a suitable η^* such that $|\eta^*|\leqslant\Lambda_{s,M}(y)$.
More explicitly, the latter quantity is equal to

$$\left|{}_y\Delta^{s}_{\eta^*}\sum_{\rho=0}^{r}(-1)^{\rho}\binom{r}{\rho}\cdot\left(\frac{d}{dy}\right)^{q}(D^{(p,0)}f)_{x^*+\rho\cdot\delta^*}(y^*)\right|$$

$$=\left|{}_y\Delta^{s}_{\eta^*}\sum_{\rho=0}^{r}(-1)^{\rho}\binom{r}{\rho}\cdot(D^{(p,q)}f)_{x^*+\rho\cdot\delta^*}(y^*)\right|$$

$$=\left|\sum_{\sigma=0}^{s}(-1)^{\sigma}\binom{s}{\sigma}\sum_{\rho=0}^{r}(-1)^{\rho}\binom{r}{\rho}\cdot(D^{(p,q)}f)(x^*+\rho\cdot\sigma^*, y^*+\sigma\eta^*)\right|$$

$$=\left|\sum_{\sigma=0}^{s}\sum_{\rho=0}^{r}(-1)^{\sigma+\rho}\binom{s}{\sigma}\binom{r}{\rho}\cdot(D^{(p,q)}f)(x^*+\rho\cdot\delta^*, y^*+\sigma\eta^*)\right|$$

$$\leqslant\omega_{r,s}(f^{(p,q)}; \Lambda_{r,L}(x), \Lambda_{s,M}(y)).$$

Combining the last inequality with the observations made earlier shows the validity of the theorem. ∎

3. On Univariate Cubic Spline Interpolation

There are a number of articles dealing with the rate of convergence of univariate Type I spline interpolators (When speaking of the convergence of cubic spline interpolation, one envisions a sequence of meshes Δ_n such that the mesh gauges $\delta = \delta_n \to 0$ as $n \to \infty$). For the sake of brevity we will write $S_n := S_{\Delta_n}$ in this section only. The following lemma summarizes various results; it is taken from papers of Hall [24], Carlson and Hall [8], and Hall and Meyer [27].

Lemma 3.1. *Let S_n be given as above, and $p = 1, 2, 3,$ or 4. Then for all $f \in C^p(I)$, there holds*

$$\|(S_n f - f)^{(k)}\|_\infty \leqslant \varepsilon_{p,k} \cdot \delta^{p-k} \cdot \|f^{(p)}\|_\infty, \qquad 0 \leqslant k \leqslant \min\{p, 3\}.$$

Here $\|\cdot\|_\infty$ is the sup norm, and the $\varepsilon_{p,k}$ are given by the following table (β denotes the mesh ratio of Δ_n)

$\varepsilon_{p,k}$	$k=0$	$k=1$	$k=2$	$k=3$
$p=1$	15/4	14	—	—
$p=2$	9/8	4	10	—
$p=3$	71/216	31/27	5	$(63+8\beta^2)/9$
$p=4$	5/384	1/24	3/8	$(\beta+\beta^{-1})/2$

Since $S_n f \in C^2(I)$ only, for the case $k=3$, the above inequality is intended to express the fact that both the right and left derivatives satisfy it at the points of discontinuity (see [8]).

Another type of inequality was already given in 1966 by Sharma and Meir [45] (see also Müller [35]). Using ω_1, the first order modulus of continuity, given for $\delta \geqslant 0$ by

$$\omega_1(f; \delta) = \sup\{|f(x') - f(x'')| : |x' - x''| \leqslant \delta\},$$

they proved

Theorem 3.2. *For S_n as above, the following holds for any $f \in C^2[0, 1]$:*

$$\|(S_n f - f)^{(k)}\|_\infty \leqslant 5 \cdot \delta^{2-k} \cdot \omega_1(f'', \delta), \qquad 0 \leqslant k \leqslant 2.$$

A certain supplement of the above result is due to Schmidt [42], who showed that for $f \in C^3[0, 1]$,

$$\|(S_n f - f)^{(k)}\|_\infty \leqslant (9/4) \cdot \delta^{3-k} \cdot \omega_1(f''', \delta), \qquad 0 \leqslant k \leqslant 2.$$

This result was further supplemented in a paper of Gfrerer [11]. He showed that for $f \in C^3[0, 1]$, one also has

$$\|(S_n f - f)^{(3)}\|_\infty \leqslant M \cdot \omega_1(f''', \delta),$$

where M is a constant depending only on two positive real numbers K and $q < \frac{1}{2} \cdot (3 + \sqrt{5})$ such that for all $n \in \mathbb{N}$ one has

$$\Delta x_i / \Delta x_j \leqslant K \cdot q^{|i-j|}, \qquad 0 \leqslant i, j \leqslant n-1.$$

The aim of the present section is to show that the theorem of Sharma and Meir can be further extended by using univariate *moduli of smoothness of higher order*. For $s \in \mathbb{N}_0$, $f \in C(I)$, and $\delta \geqslant 0$, these functionals are given by

$$\omega_s(f, \delta) := \sup \left\{ \left| \sum_{\nu=0}^{s} (-1)^{s-\nu} \binom{s}{\nu} f(x + \nu h) \right| : x, x + sh \in I, |h| \leqslant \delta \right\}.$$

For various properties of ω_s, see Schumaker's book [44; p. 55ff, and the references cited therein]. The crucial tool needed to achieve such estimates is given by the following

LEMMA 3.3 (Gonska [13]). *Let $I = [0, 1]$ and $f \in C^r(I)$, $r \in \mathbb{N}_0$. For any $\delta \in (0, 1]$ and $s \in \mathbb{N}$, there exists a function $g_{\delta, r+s} \in C^{2r+s}(I)$ with*

(i) $\|f^{(j)} - g^{(j)}_{\delta, r+s}\|_\infty \leqslant c \cdot \omega_{r+s}(f^{(j)}, \delta)$ *for* $0 \leqslant j \leqslant r$.

(ii) $\|g^{(j)}_{\delta, r+s}\|_\infty \leqslant c \cdot \delta^{-j} \cdot \omega_j(f, \delta)$ *for* $0 \leqslant j \leqslant r + s$.

Here, the constant c depends only on r and s.

The extension of the result of Sharma and Meir is now as follows.

THEOREM 3.4. *Let S_n be given as above, and let $p = 1, 2, 3$, or 4. Then for any $f \in C^p(I)$ the following inequalities hold:*

$$\|(S_n f - f)^{(k)}\|_\infty \leqslant c(p, k) \cdot \delta^{p-k} \cdot \omega_{4-p}(f^{(p)}, \delta), \qquad 0 \leqslant k \leqslant \min\{p, 3\}.$$

Here, the constants $c(p, k)$ depend only on p and k, and, for $k = 3$, also on the mesh ratio β.

Proof. Let $f \in C^p(I)$ and $0 \leqslant k \leqslant \min\{p, 3\}$. For any $g \in C^4(I)$, we have

$$\|(S_n f - f)^{(k)}\|_\infty \leqslant \|(S_n(f-g) - (f-g))^{(k)}\|_\infty + \|(S_n g - g)^{(k)}\|_\infty .$$

Using the constants $\varepsilon_{p,k}$ from Lemma 3.1, the first summand is bounded from above by

$$\varepsilon_{p,k} \cdot \|(f-g)^{(p)}\|_\infty \cdot \delta^{p-k}.$$

Likewise,

$$\|(S_n g-g)^{(k)}\|_\infty \leqslant \varepsilon_{4,k}\cdot\|g^{(4)}\|\cdot\delta^{4-k}.$$

Hence,

$$\begin{aligned}&\|(S_n f-f)^{(k)}\|_\infty\\ &\quad\leqslant \max\{\varepsilon_{p,k};\varepsilon_{4,k}\}\cdot\delta^{p-k}\cdot\{\|(f-g)^{(p)}\|_\infty+\delta^{4-p}\cdot\|g^{(4)}\|_\infty\}.\end{aligned}$$

For $p=4$, we can take $g=f$ to obtain the original inequality

$$\|(S_n f-f)^{(k)}\|_\infty\leqslant\varepsilon_{4,k}\cdot\delta^{4-k}\cdot\|f^{(4)}\|_\infty,\qquad 0\leqslant k\leqslant 3.$$

If $p<4$, it follows from Lemma 3.3(i) [use $r=0$ and $s=4-p$ there] that there is a function $g_{\delta,0+4-p}\in C^{4-p}(I)$ such that

$$\|f^{(p)}-g_{\delta,4-p}\|_\infty\leqslant c\cdot\omega_{4-p}(f^{(p)},\delta).$$

If G_δ denotes a pth primitive of $g_{\delta,4-p}$, then $G_\delta\in C^4(I)$ and the latter inequality becomes

$$\|f^{(p)}-G_\delta^{(p)}\|_\infty\leqslant c\cdot\omega_{4-p}(f^{(p)},\delta).$$

Furthermore, Lemma 3.3(ii) guarantees that

$$\begin{aligned}\|G_\delta^{(4)}\|_\infty&=\|G_\delta^{(p+4-p)}\|_\infty\\ &=\|g_\delta^{(4-p)}\|_\infty\\ &\leqslant c\cdot\delta^{-(4-p)}\cdot\omega_{4-p}(f^{(p)},\delta).\end{aligned}$$

Hence,

$$\begin{aligned}&\|(S_n f-f)^{(k)}\|_\infty\\ &\quad\leqslant\max\{\varepsilon_{p,k};\varepsilon_{4,k}\}\cdot\delta^{p-k}\cdot\{\|(f-G_\delta)^{(p)}\|_\infty+\delta^{4-p}\cdot\|G_\delta^{(4)}\|_\infty\}\\ &\quad\leqslant c\cdot\max\{\varepsilon_{p,k};\varepsilon_{4,k}\}\cdot\delta^{p-k}\\ &\qquad\cdot\{\omega_{4-p}(f^{(p)},\delta)+\delta^{4-p}\cdot\delta^{-(4-p)}\cdot\omega_{4-p}(f^{(p)},\delta)\}\\ &\quad=2c\cdot\max\{\varepsilon_{p,k};\varepsilon_{4,k}\}\cdot\delta^{p-k}\cdot\omega_{4-p}(f^{(p)},\delta),\end{aligned}$$

and thus the inequality is also proved for $1\leqslant p\leqslant 3$. ∎

Remark 3.5. (i) Because

$$\omega_{4-p}(f^{(p)};\delta)\leqslant c\cdot\|f^{(4)}\|_\infty,\qquad f\in C^4(I),$$

the inequality of Theorem 3.4 expresses the fact that each cubic polynomial is reproduced by $\check{S}_n$.

(ii) For the case $p=2$, Theorem 3.4 gives

$$\|(S_n f-f)^{(k)}\|_\infty \leqslant c(2,k)\cdot\delta^{2-k}\cdot\omega_2(f'',\delta), \qquad 0\leqslant k\leqslant 2,$$

which is an improvement of the result of Sharma and Meir [45].

(iii) For $p=3$, we have

$$\|(S_n f-f)^{(k)}\|_\infty \leqslant c(3,k)\cdot\delta^{3-k}\cdot\omega_1(f''',\delta), \qquad 0\leqslant k\leqslant 3.$$

This inequality is a combination of the results of Schmidt and Gfrerer mentioned earlier (see [42, 11]).

(iv) The choice $p=4$ gives

$$\|(S_n f-f)^{(k)}\|_\infty \leqslant c(4,k)\cdot\delta^{4-k}\cdot\|f^{(4)}\|_\infty, \qquad 0\leqslant k\leqslant 3.$$

In particular, we have for $0\leqslant k\leqslant 2$ that

$$\|(S_n f-f)^{(k)}\|_\infty = O(\delta^{4-k}), \qquad \delta\to 0,$$

independent of the mesh radio β. (In the latter cases the constants $c(4,k)= 2c\cdot\varepsilon_{4,k}$, $0\leqslant k\leqslant 2$, do not depend on β.)

4. Simultaneous Approximation by Gordon Operators

4.1. *Blended Type* I *Spline Interpolators*

THEOREM 4.1. *Let* $p=1,2,3,$ *or* 4 *and* $S_{\Delta_n}: C^p(I)\ni f\mapsto s_f\in C^2(I)$ *be the "clamped spline" operator from Theorem* 3.4, *i.e.*,

$$\|(g-S_{\Delta_n} g)^{(k)}\|_\infty \leqslant c(p,k)\cdot\delta^{p-k}\cdot\omega_{4-p}(f^{(p)},\delta),$$
$$0\leqslant k\leqslant p^* := \min\{p,2\}\leqslant \min\{p,3\}.$$

Let $q=1,2,3,$ *or* 4, *and analogously* $S_{\Delta_m}: C^q(J)\to C^2(J)$ *be given with*

$$\|(h-S_{\Delta_m} h)^{(l)}\|_\infty \leqslant c(q,l)\cdot\tilde\delta^{q-l}\cdot\omega_{4-q}(f^{(q)},\tilde\delta),$$
$$0\leqslant l\leqslant q^* := \min\{q,2\}\leqslant \min\{q,3\},$$

where $\tilde\delta$ *is the mesh gauge of* Δ_m.
Then we have for $(0,0)\leqslant(k,l)\leqslant(p^*,q^*)$ *and all* $f\in C^{p,q}(I\times J)$

$$\|(f-({}_xS_{\Delta_n}\oplus{}_yS_{\Delta_m})f)^{(k,l)}\|_\infty$$
$$\leqslant c(p,k)\cdot c(q,l)\cdot\delta^{p-k}\cdot\tilde\delta^{q-l}\cdot\omega_{4-p,4-q}(f^{(p,q)};\delta,\tilde\delta).$$

The proof is a consequence of Theorem 2.2.

COROLLARY 4.2. (i) *For* $p=q=1$, *we obtain with* $0 \leqslant k, l \leqslant 1$

$$\|(f-({}_xS_{\Delta_n} \oplus {}_yS_{\Delta_m})f)^{(k,l)}\|_\infty \leqslant c(1,k) \cdot c(1,l) \cdot \delta^{1-k} \cdot \tilde{\delta}^{1-l} \cdot \omega_{3,3}(f^{(1,1)}; \delta, \tilde{\delta}).$$

(ii) *For* $p=q=4$, *we get for* $0 \leqslant k, l \leqslant 2$

$$\begin{aligned}\|(f-({}_xS_{\Delta_n} \oplus {}_yS_{\Delta_m})f)^{(k,l)}\|_\infty &\leqslant c(4,k) \cdot c(4,l) \cdot \delta^{4-k} \cdot \tilde{\delta}^{4-l} \cdot \|f^{(4,4)}\|_\infty \\ &= O(\delta^{4-k} \cdot \tilde{\delta}^{4-l}), \quad \delta, \tilde{\delta} \to 0,\end{aligned}$$

(*see Carlson–Hall* [8, *Theorem* 2]).

4.2. *Blended Natural Spline Interpolators*

LEMMA 4.3 (Hall [25, Corollary 2]). *If the natural cubic spline operator with respect to the partition* $\Delta_n: x_0 < \cdots < x_n$ *is denoted by* T_{Δ_n}, *then for all* $g \in C^2[x_0, x_n]$ *and* $k=0, 1$ *there holds*

$$\|(T_{\Delta_n}g-g)^{(k)}\|_\infty \leqslant c \cdot \delta^{2-k} \cdot \|g''\|_\infty \quad (=c \cdot \delta^{2-k} \cdot \omega_0(g'', \delta)).$$

Here, δ *is the mesh gauge of* Δ_n, *and* c *is a constant independent of* Δ_n.

In the following theorem we give an inequality on the degree of simultaneous approximation by Gordon operators based upon the use of T_{Δ_n}. Here, T_{Δ_n} is considered as an operator mapping $C^2(I)$ into $C^1(I)$. $C^1(I)$ was chosen as the image space in order to indicate that uniform simultaneous approximation by blended *natural* spline interpolators can be expected only for mixed partials of order $(k, l) \leqslant (1, 1)$.

THEOREM 4.4. *Let* $T_{\Delta_n}: C^2(I) \to C^1(I)$ *be the natural cubic spline operator from Lemma* 4.3. *Analogously, let* $T_{\Delta_m}: C^2(J) \to C^1(J)$ *be given, with*

$$\|(T_{\Delta_m}h-h)^{(l)}\|_\infty \leqslant \tilde{c} \cdot \tilde{\delta}^{2-l} \cdot \|h''\|_\infty \quad \text{for} \quad l=0, 1,$$

where $\tilde{\delta}$ *is the mesh gauge of* Δ_m. *Then for* $(0,0) \leqslant (k,l) \leqslant (1,1)$ *and all* $f \in C^{2,2}(I \times J)$, *we have*

$$\|(f-({}_xT_{\Delta_n} \oplus {}_yT_{\Delta_m})f)^{(k,l)}\|_\infty \leqslant c \cdot \tilde{c} \cdot \delta^{2-k} \cdot \tilde{\delta}^{2-l} \cdot \|f^{(2,2)}\|_\infty.$$

The proof follows from Theorem 2.2.

5. Numerical Approximation

This section is concerned with the question of the numerical implementation of the blending schemes discussed so far, and with the degree of approximation by *discrete blended interpolants* (*approximants*). The general idea to construct such discrete schemes was already described by Gordon [19, p. 254]: "First into univariate functions and then into scalar parameters." It was further discussed by Cavendish, Gordon, and Hall [23, 9] and Lancaster [28]. Formally, the approach is as follows:

let $l_y, u_y \in \mathbb{Z}$ be such that $l_y \leqslant u_y$, and, for $l_y \leqslant j \leqslant u_y$, let $\tau_j: C^q(J) \to \mathbb{R}$ be given linear functionals so that for all $f \in C^{p,q}(I \times J)$ the function $[I \ni x \mapsto \tau_j(f_x)]$ is in $C^p(I)$. If the operator $M: C^q(J) \to C^{q'}(J)$ (see Theorem 2.2) has the form

$$M(f; y) = \sum_{j=l_y}^{u_y} \tau_j(f) \cdot h_j(y), \qquad h_j \in C^{q'}(J),\ l_y \leqslant j \leqslant u_y,$$

then its parametric extension ${}_yM: C^{p,q}(I \times J) \to C^{p,q'}(I \times J)$ is defined by

$${}_yM(f; x, y) = M(f_x, y) = \sum_{j=l_y}^{u_y} \tau_j(f_x) \cdot h_j(y),$$

where f_x denotes the partial functions of f. Hence, in general, ${}_yM$ has infinite dimensional range.

Using this notation, the parametric extension ${}_yS_{\Delta_m}$ of the clamped cubic spline interpolation operator S_{Δ_m} discussed above is obtained by choosing $l_y = -1$, $u_y = m+1$, and

$$\tau_j(f) = \begin{cases} f'(y_0), & j = -1 \\ f(y_j), & 0 \leqslant j \leqslant m \\ f'(y_m), & j = m+1. \end{cases}$$

Hence, in order to represent ${}_yM(f)$ numerically, it is necessary to find an approximation of the x-dependence of this function. This can be achieved by applying an operator ${}_x\bar{L}: C^{p,q'}(I \times J) \to C^{p',q'}(I \times J)$ of the form

$${}_x\bar{L}(f; x, y) = \sum_{i=\bar{l}_x}^{\bar{u}_x} \bar{\lambda}_i(f^y) \cdot \bar{g}_i(x), \qquad \bar{g}_i \in C^{p'}(I),\ \bar{l}_x \leqslant i \leqslant \bar{u}_x,$$

to ${}_yM(f)$, where $\bar{l}_x, \bar{u}_x \in \mathbb{Z}$, $\bar{l}_x \leqslant \bar{u}_x$, and where the functionals $\bar{\lambda}_i: C^p(I) \to \mathbb{R}$ now act on the variable $x \in I$ and are such that for all $f \in C^{p,q'}(I \times J)$ the function $[J \ni y \mapsto \bar{\lambda}_i(f^y)]$ is in $C^{q'}(J)$. The resulting function $({}_x\bar{L} \circ {}_y M)f$ is then given by

$$({}_x\bar{L} \circ {}_y M)(f; x, y) = \sum_{i=\bar{l}_x}^{\bar{u}_x} \sum_{j=l_y}^{u_y} \bar{\lambda}_i \tau_j[f] \cdot \bar{g}_i(x) \cdot h_j(y),$$

and is an element of $C^{p', q'}(I \times J)$.

In the case where both $\bar{L}$ and M are cubic spline operators, ${}_x\bar{L} \circ {}_y M$ is their tensor product, a bicubic spline operator. See Mettke [33] for a more detailed explanation and for further examples.

Recalling that in Section 2 we were considering approximants of the form ${}_xL \oplus {}_yM$, it is appropriate to make a note on the relationship between ${}_xL$ and ${}_x\bar{L}$. Usually the univariate operator $\bar{L}$ is obtained by "refining" L in a certain fashion so that ${}_x\bar{L}$ provides much better approximations to bivariate functions than ${}_xL$ does. The corresponding discretizations ${}_y\bar{M} \circ {}_xL$ are obtained in an analogous way.

The operator P being used for numerical approximation is then given by

$$P := {}_y\bar{M} \circ {}_xL + {}_x\bar{L} \circ {}_yM - {}_xL \circ {}_yM,$$

and is sometimes called the *discrete* $(L, M, \bar{L}, \bar{M})$ *blending interpolation* (*approximation*) *operator*. For the general technique of "nth order blending" (in connection with the so-called complete interpolation scheme for polynomial splines) see, e.g., Baszenski [4]; the above operator P is related to second order blending.

A fundamental lemma upon which the considerations of this section will be based was given by Gordon [19, Lemma A1] (see also Hall [26, Theorem 2, p. 330ff].

LEMMA 5.1. *Let* $(\mathscr{A}, +, 0)$ *be a (not necessarily commutative) ring with unity* I, $\mathscr{A} = \{I, A, B, C, C, ...,\}$. *If one defines the Boolean sum of* $A, B \in \mathscr{A}$ *by* $A \oplus B := A + B - A \circ B$, *and if* $A \circ D = D \circ A$, *then*

$$\begin{aligned} I - (D \circ A + C \circ B - A \circ B) \\ = (I - C) + (I - D) + (I - A \oplus B) - (I - C \oplus B) - (I - A \oplus D). \end{aligned}$$

Proof. Writing out the right hand side explicitly and observing that $A \circ D = D \circ A$ yields the claim of the lemma. ∎

THEOREM 5.2. *Let univariate operators L and M be given as in Theorem 2.2. Furthermore, let* $\bar{L}: C^p(I) \to C^{p'}(I)$ *be given such that*

$$|(g - \bar{L}g)^{(k)}(x)| \leqslant \Gamma_{r,k,\bar{L}}(x) \cdot \omega_r(g^{(p)}; \Delta_{r,\bar{L}}(x)),$$

where $\Gamma_{r,k,\bar{L}}(x) \leqslant \Gamma_{r,k,L}(x)$, $\Delta_{r,\bar{L}}(x) \leqslant \Delta_{r,L}(x)$ *for all* $x \in I$, *all* $g \in C^p(I)$, *and all* $0 \leqslant k \leqslant p^* = \min\{p, p'\}$.

Also, let $\bar{M}: C^q(J) \to C^{q'}(J)$ *be such that*

$$|(h - \bar{M}h)^{(l)}(y)| \leqslant \Gamma_{s,l,\bar{M}}(y) \cdot \omega_s(h^{(q)}; \Lambda_{s,\bar{M}}(y)),$$

where $\Gamma_{s,l,M}(y) \leqslant \Gamma_{s,l,\bar{M}}(y)$, $\Lambda_{s,M}(y) \leqslant \Lambda_{s,\bar{M}}(y)$ *for* $y \in J$, *all* $h \in C^q(J)$, *and all* $0 \leqslant l \leqslant q^* = \min\{q, q'\}$.

The Γ's and Λ's are assumed to be bounded real-valued functions. Then for all $(x, y) \in I \times J$ *and all* $f \in C^{p,q}(I \times J)$, *we have for* $(0, 0) \leqslant (k, l) \leqslant (p^*, q^*)$:

$$\begin{aligned} &|D^{(k,l)} \circ (I - {}_y\bar{M} \circ {}_xL - {}_x\bar{L} \circ {}_yM + {}_xL \circ {}_yM)(f; x, y)| \\ &\quad \leqslant \Gamma_{r,k,\bar{L}}(x) \cdot \omega_{r,0}(f^{(p,l)}; \Lambda_{r,\bar{L}}(x), 0) \\ &\qquad + \Gamma_{s,l,\bar{M}}(y) \cdot \omega_{0,s}(f^{(k,q)}; 0, \Lambda_{s,\bar{M}}(y)) \\ &\qquad + 3 \cdot \Gamma_{r,k,L}(x) \cdot \Gamma_{s,l,M}(y) \cdot \omega_{r,s}(f^{(p,q)}; \Lambda_{r,L}(x), \Lambda_{s,M}(y)). \end{aligned}$$

Proof. We decompose the difference $I - ({}_y\bar{M} \circ {}_xL + {}_x\bar{L} \circ {}_yM - {}_xL \circ {}_yM)$ as suggested by Lemma 5.1 and apply $D^{(k,l)}$. This yields the five term expression

$$\begin{aligned} &D^{(k,l)} \circ (I - {}_x\bar{L}) + D^{(k,l)} \circ (I - {}_y\bar{M}) + D^{(k,l)} \circ (I - {}_xL \oplus {}_yM) \\ &\quad - D^{(k,l)} \circ (I - {}_x\bar{L} \oplus {}_yM) - D^{(k,l)} \circ (I - {}_xL \oplus {}_y\bar{M}). \end{aligned}$$

The first term can be further decomposed as

$$\begin{aligned} D^{(k,l)} \circ (I - {}_x\bar{L}) &= D^{(k,0)} \circ D^{(0,l)} \circ (I - {}_x\bar{L}) \\ &= D^{(k,0)} \circ (I - {}_x\bar{L}) \circ D^{(0,l)}. \end{aligned}$$

Here we have used the fact that for $0 \leqslant l \leqslant q$ the operators $D^{(0,l)}$ and ${}_x\bar{L}$ commute on $C^{p,q}(I \times J)$ (see Lemma 2.1). The assumption on $\bar{L}$ implies

$$\begin{aligned} &|(D^{(k,l)} \circ (I - {}_x\bar{L}))(f; x, y)| \\ &\quad \leqslant \Gamma_{r,k,L}(x) \cdot \omega_{r,0}(f^{(p,l)}; \Lambda_{r,\bar{L}}(x), 0), \qquad 0 \leqslant k \leqslant p^*. \end{aligned}$$

In a similar way, it follows for $0 \leqslant k \leqslant p$ that

$$\begin{aligned} &|(D^{(k,l)} \circ (I - {}_y\bar{M}))(f; x, y)| \\ &\quad \leqslant \Gamma_{s,l,\bar{M}}(y) \cdot \omega_{0,s}(f^{(k,q)}; 0, \Lambda_{s,\bar{M}}(y)), \qquad 0 \leqslant l \leqslant q^*. \end{aligned}$$

The remaining three terms have upper bounds as suggested by Theorem 2.2. We obtain for $(0, 0) \leqslant (k, l) \leqslant (p^*, q^*)$ the inequalities

$$\begin{aligned} &|(D^{(k,l)} \circ (I - {}_xL \oplus {}_yM))(f; x, y)| \\ &\quad \leqslant \Gamma_{r,k,L}(x) \cdot \Gamma_{s,l,M}(y) \cdot \omega_{r,s}(f^{(p,q)}; \Lambda_{r,L}(x), \Lambda_{s,M}(y)), \end{aligned}$$

$$
\begin{aligned}
&|(D^{(k,l)} \circ (I - {}_x\bar{L} \oplus {}_y M))(f; x, y)| \\
&\quad \leqslant \Gamma_{r,k,\bar{L}}(x) \cdot \Gamma_{s,l,M}(y) \cdot \omega_{r,s}(f^{(p,q)}; \Lambda_{r,\bar{L}}(x), \Lambda_{s,M}(y)) \\
&\quad \leqslant \Gamma_{r,k,L}(x) \cdot \Gamma_{s,l,M}(y) \cdot \omega_{r,s}(f^{(p,q)}; \Lambda_{r,L}(x), \Lambda_{s,M}(y)), \\
&|(D^{(k,l)} \circ (I - {}_x L \oplus {}_y \bar{M}))(f; x, y)| \\
&\quad \leqslant \Gamma_{r,k,L}(x) \cdot \Gamma_{s,l,\bar{M}}(y) \cdot \omega_{r,s}(f^{(p,q)}; \Lambda_{r,L}(x), \Lambda_{s,\bar{M}}(y)) \\
&\quad \leqslant \Gamma_{r,k,L}(x) \cdot \Gamma_{s,l,M}(y) \cdot \omega_{r,s}(f^{(p,q)}; \Lambda_{r,L}(x), \Lambda_{s,M}(y)).
\end{aligned}
$$

Combining the five inequalities yields the claim of Theorem 5.2. ∎

Remark 5.3. (i) Theorem 5.2 is not given in its most general form. It is, for instance, possible to assume that $\bar{L}$ satisfies certain inequalities in terms of $\omega_{\bar{r}}$ instead of ω_r. Furthermore, assumption such as $\Gamma_{r,k,\bar{L}}(x) \leqslant \Gamma_{r,k,L}(x)$, etc., were mainly made in order to obtain simple upper bounds of the differences in question. We note, however, that our simplifying assumptions do not have a negative impact on the order of convergence in the application below.

(ii) Another way to assume compatibility between L and $\bar{L}$, and between M and $\bar{M}$, respectively, is to require, for instance, absorption properties such as $\bar{L} \circ L = L$ and $\bar{M} \circ M = M$ to hold. This approach is quite typical for the case of spline projectors and was discussed to some extent in [9, 28].

In the following example, we apply Theorem 5.2 to discrete Type I (clamped) spline blended interpolation.

EXAMPLE 5.4. For the sake of simplicity, in the sequel, c will always denote a suitable numerical constant which may be different at different occurrences, even on the same line.

Let $S_{\Delta_n}: C^p(I) \to C^2(I)$ be given as in Theorem 3.4, i.e., $p = 1, 2, 3$, or 4, and

$$\|(S_{\Delta_n} f - f)^{(k)}\|_\infty \leqslant c \cdot \delta^{p-k} \cdot \omega_{4-p}(f^{(p)}; \delta), \qquad 0 \leqslant k \leqslant \min\{p, 3\}.$$

In addition, let $\delta \leqslant 1$.

Let $S_{\Delta_m}: C^q(J) \to C^2(J)$ be a second spline operator of the same type (and thus satisfying an analogous inequality involving the mesh gauge $\tilde{\delta} \leqslant 1$).

Choose further an operator

$$\bar{S}_{\Delta_{\bar{n}}}: C^p(I) \to C^2(I)$$

with respect to the mesh $\Delta_{\tilde{n}}$ and mesh gauge δ^2, as well as another operator

$$S_{\Delta_{\tilde{m}}}: C^q(J) \to C^2(J)$$

with mesh $\Delta_{\tilde{m}}$ and gauge $\tilde{\delta}^2$. Using the notation

$$P_{\delta,\tilde{\delta}} := {}_yS_{\Delta_{\tilde{m}}} \circ {}_xS_{\Delta_n} + {}_xS_{\Delta_{\tilde{n}}} \circ {}_yS_{\Delta_m} - {}_xS_{\Delta_n} \circ {}_yS_{\Delta_m},$$

from Theorem 5.2, we arrive at

$$\begin{aligned}
&\| D^{(k,l)} \circ (I - P_{\delta,\tilde{\delta}}) f \|_\infty \\
&\quad \leqslant c \cdot (\delta^2)^{p-k} \cdot \omega_{4-p,0}(f^{(p,l)}; \delta^2, 0) + c \cdot (\tilde{\delta}^2)^{q-l} \cdot \omega_{0,4-q}(f^{(k,q)}; 0, \tilde{\delta}^2) \\
&\qquad + c \cdot \delta^{p-k} \cdot \tilde{\delta}^{q-l} \cdot \omega_{4-p,4-q}(f^{(p,q)}; \delta, \tilde{\delta}), \\
&\qquad \text{for } (0,0) \leqslant (k,l) \leqslant (\min\{p,2\}, \min\{q,2\}).
\end{aligned}$$

(i) For $p = q = 1$ (hence $p^* = q^* = 1$) and $\delta = \tilde{\delta}$, the upper bound from above becomes

$$\begin{aligned}
&c \cdot \delta^{2(1-k)} \cdot \omega_{3,0}(f^{(1,l)}; \delta^2, 0) \\
&\quad + c \cdot \delta^{2(1-l)} \cdot \omega_{0,3}(f^{(k,1)}; 0, \delta^2) \\
&\quad + c \cdot \delta^{2-k-l} \cdot \omega_{3,3}(f^{(1,1)}; \delta, \delta), \qquad (0,0) \leqslant (k,l) \leqslant (1,1).
\end{aligned}$$

For $f \in C^{1,1}(I \times J)$, the latter quantity is bounded from above by

$$\begin{aligned}
&c \cdot \delta^{2-2k} \cdot \| f^{(1,l)} \| + C \cdot \delta^{2-2l} \cdot \| f^{(k,1)} \| + c \cdot \delta^{2-k-l} \cdot \| f^{(1,1)} \| \\
&\quad = O(\delta^{2-2\max\{k,l\}}), \qquad \delta \to 0.
\end{aligned}$$

This implies $O(\delta^2)$, $\delta \to 0$, convergence of $P_{\delta,\delta} f$ to f. However, it does not imply uniform convergence of either of the derivatives $(P_{\delta,\delta} f)^{(k,l)}$ to $f^{(k,l)}$ for $k = 1$ or $l = 1$. Non-quantitative assertions of this type are obtained by referring back to the previous three term expression for the case $p = q = 1$: all three moduli present there tend to zero as $\delta \to 0$.

Furthermore, for $f \in C^{4,4}(I \times J)$, the upper bound is

$$\begin{aligned}
&c \cdot \delta^{2(1-k)} \cdot \delta^6 \cdot \| f^{(4,l)} \|_\infty + c \cdot \delta^{2(1-l)} \cdot \delta^6 \cdot \| f^{(k,4)} \|_\infty \\
&\quad + c \cdot \delta^{2-k-l} \cdot \delta^6 \cdot \| f^{(4,4)} \|_\infty \\
&\quad = O(\delta^{8-2\max\{k,l\}}), \qquad \delta \to 0, \ (0,0) \leqslant (k,l) \leqslant (1,1).
\end{aligned}$$

An inspection of the possible cases for k, l shows that any of the three $O(\cdots)$ terms may be asymptotically dominant.

(ii) For $p=q=4$ (hence $p^*=q^*=2$) and $\tilde{\delta}=\delta$, we obtain

$$\begin{aligned}
&\| D^{(k,l)} \circ (I-P_{\delta,\delta}) f \|_\infty \\
&\quad \leqslant c \cdot (\delta^2)^{4-k} \cdot \| f^{(4,l)} \|_\infty + c \cdot (\delta^2)^{4-l} \cdot \| f^{(k,4)} \|_\infty \\
&\qquad + c \cdot \delta^{8-k-l} \cdot \| f^{(4,4)} \|_\infty \\
&\quad = O(\delta^{8-2\max\{k,l\}}) \qquad \text{for} \quad \delta \to 0 \qquad \text{and} \qquad (0,0) \leqslant (k,l) \leqslant (2,2).
\end{aligned}$$

This is, of course, the same order of approximation as obtained under (i). However, now statements for the degree of simultaneous approximation with respect to $D^{(2,l)}$, $0 \leqslant l \leqslant 2$, and $D^{(k,2)}$, $0 \leqslant k \leqslant 2$, are also available. In all five cases just mentioned we derive

$$\| D^{(k,l)} \circ (I-P_{\delta,\delta}) f \|_\infty = O(\delta^4), \qquad \delta \to 0.$$

Note that, as far as order is concerned, our estimates are similar to the L_2 norm inequalities derived in [9, Th. 3] for functions in the Sobolev space $W_2^{(4,4)}(I \times J)$. It should be noted that they were able to prove a corresponding statement for $(0,0) \leqslant (k,l) \leqslant (3,3)$. However, our estimate from above is only one particular instance of the much more general approach described in Theorem 5.2 and of its particular consequence given just before this example. ∎

Remark 5.5. Inequalities analogous to those of Example 5.4 are available for the discrete version of the blended natural spline interpolation operator discussed in Section 4.2.

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